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Calculus of Variations

ERRATUM

Rank-one convex hulls in $\mathbb{R}^{2\times 2}$

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In the proof of the separation theorem (Theorem 4) in the paper *Rank-one convex hulls in* $\mathbb{R}^{2\times 2}$ (Calc. Var. **22**, 2005) an important ingredient is that under the assumption that for any $X_1, X_2 \in K_1$ and $Y_1, Y_2 \in K_2$ the set $\{X_1, X_2, Y_1, Y_2\}$ does not form a T_4 configuration, the sets $E(X_1, X_2, Y_1) \setminus \{y_1^+\}$ and $E(X_1, Y_1, Y_2) \setminus \{x_1^+\}$ are disjoint. This is proved in Lemma 7, where we use the claim of Lemma 6 (v) that

$$E(X_1, X_2, X_3) = \mathcal{E} \cup \{x_1^+, x_2^+, x_3^+\} \tag{1}$$

for some solid ellipse $\mathcal{E}$ (meaning a closed convex set whose boundary is an ellipse). In fact this claim is false, but for the proof of Lemma 7 (and Theorem 4) it suffices if (1) holds for some **connected** set $\mathcal{E}$. In this note we intend to prove this and thereby correct this mistake in the paper. Recall that for any $X_1, X_2, X_3 \in \mathbb{R}^{2\times 2}$ we define

$$E(X_1, X_2, X_3) \stackrel{\text{def}}{=} \left\{ z \in \mathbb{C} : \bigcap_{i=1}^{3} B_{|x_i^+ - z|}(x_i^-) = \emptyset \right\},$$

where we write 2×2 matrices X in conformal-anticonformal coordinates as $X = (x^+, x^-)$ for $x^+, x^- \in \mathbb{C}$ so that $\det X = |x^+|^2 - |x^-|^2$. In order to keep the notation simple, we write

$$B_z^i = B_{|x_i^+ - z|}(x_i^-) \quad \text{and} \quad E = E(X_1, X_2, X_3).$$

Lemma 1 *For any X_1, X_2, X_3 there exists a connected set $\mathcal{E} \subset \mathbb{C}$ such that*

$$E = \mathcal{E} \cup \{x_1^+, x_2^+, x_3^+\}.$$

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Moreover if $\det(X_i - X_j) > 0$ *for each* i,j, *then* $E = \{x_1^+, x_2^+, x_3^+\}$.

Proof 1 First of all it is clear that if $B_z^i \cap B_z^j = \emptyset$ for some $z \in \mathbb{C}$ and some $i,j \in \{1,2,3\}$, then $z \in E$. The set of such z is given by

$$\mathcal{E}_{ij} = \{z \in \mathbb{C} : |z - x_i^+| + |z - x_j^+| \leq |x_i^- - x_j^-|\},$$

which is either a solid ellipse (if $\det(X_i - X_j) < 0$) with focal points x_i^+, x_j^+, the line segment $[x_i^+, x_j^+]$ (if $\det(X_i - X_j) = 0$) or empty (if $\det(X_i - X_j) > 0$). Therefore the set

$$E_0 \stackrel{\text{def}}{=} \mathcal{E}_{12} \cup \mathcal{E}_{13} \cup \mathcal{E}_{23} \cup \{x_1^+, x_2^+, x_3^+\} \tag{2}$$

is contained in E. Note furthermore that since $x_i^+, x_j^+ \in \mathcal{E}_{ij}$ whenever $\mathcal{E}_{ij}$ is nonempty, the union $\mathcal{E}_{12} \cup \mathcal{E}_{13} \cup \mathcal{E}_{23}$ is connected.
2. Let $T = \{x_1^-, x_2^-, x_3^-\}^{co}$ be the triangle with vertices x_i^-. If for some z the intersection $\bigcap_{i=1}^3 B_z^i$ is nonempty, then $T \subset \bigcup_{i=1}^3 B_z^i$. Consequently if z is such that there exists $w \in T$ with $w \notin \bigcup_{i=1}^3 B_z^i$, i.e.

$$|z - x_i^+| \leq |w - x_i^-| \quad \text{for all } i = 1,2,3,$$

then $\bigcap_{i=1}^3 B_z^i = \emptyset$ and hence $z \in E$. Conversely, if $z \in E \setminus E_0$, i.e. $B_z^i \cap B_z^j \neq \emptyset$ for any i,j but $\bigcap_{i=1}^3 B_z^i = \emptyset$, then there exists $w \in T$ with $|z - x_i^+| \leq |w - x_i^-|$ for all i. Moreover w can be chosen so that equality holds for any one i.
3. Let $z \in E \setminus E_0$. By renumbering X_1, X_2, X_3 if necessary, it suffices to consider the situation where either

$$\text{a) } \det(X_1 - X_2) \leq 0 \text{ and } \det(X_1 - X_3) \leq 0,$$

or

$$\text{b) } \det(X_1 - X_2) \geq 0 \text{ and } \det(X_1 - X_3) \geq 0.$$

Let $w \in T$ be such that $|z - x_i^+| \leq |w - x_i^-|$ for all i and $|z - x_1^+| = |w - x_1^-|$, and let $Y = (z, w)$. Then $\det(Y - X_i) \leq 0$ for all i and $\det(Y - X_1) = 0$. Let

$$f_i(t) = \det(tX_1 + (1-t)Y - X_i) = \det(Y - X_i + t(X_1 - Y)).$$

Since det is quadratic and $\det(X_1 - Y) = 0$, in fact f_i is linear (more precisely affine). Moreover $f_i(0) \leq 0$ and $f_i(1) = \det(X_i - X_1)$ for all i.

In case a) we have that $f_i(1) \leq 0$ for all i, and hence $f_i(t) \leq 0$ for $t \in [0,1]$. But if $t \in [0,1]$, then $tx_1^- + (1-t)w \in T$ (since $x_1^-, w \in T$), and therefore $tx_1^+ + (1-t)z \in E$.

In case b) we have that $f_i(1) \geq 0$ for all i, therefore $f_i(t) \leq 0$ for $t \leq 0$. Since $w \in T$, the line $tx_1^- + (1-t)w$ will hit the side $[x_2^-, x_3^-]$ of the triangle T, in other words there exists $t_0 \leq 0$ such that $t_0 x_1^- + (1-t_0)w \in [x_2^-, x_3^-]$. This implies in particular that $B_{\tilde{z}}^2 \cap B_{\tilde{z}}^3 = \emptyset$ for $\tilde{z} = t_0 x_1^+ + (1-t_0)z$, so that $\tilde{z} \in \mathcal{E}_{23} \subset E_0$, and also $tx_1^+ + (1-t)z \in E$ for $t \in [t_0, 0]$.

In both cases we have demonstrated that any $z \in E \setminus E_0$ can be connected to $\mathcal{E}_{12} \cup \mathcal{E}_{13} \cup \mathcal{E}_{23}$ via a line segment inside E. This proves that there exists a connected set $\mathcal{E}$ such that $E = \mathcal{E} \cup \{x_1^+, x_2^+, x_3^+\}$.
4. Finally consider the situation when $\det(X_i - X_j) > 0$ for all i,j (with $i \neq j$). Then in particular $\mathcal{E}_{ij} = \emptyset$ for each i,j. If there exists $z \in E \setminus E_0$, then we can proceed as in case b) above to discover that there exists also $\tilde{z} \in \mathcal{E}_{23} \subset E_0$. This contradiction implies that $E = \{x_1^+, x_2^+, x_3^+\}$, thus finishing the proof of the claim. □